Combining and Contrasting Formal Concept Analysis and APOS Theory *

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Abstract. This paper investigates how two different theories (FCA and APOS Theory) complement each other with respect to applications in mathematics education research. APOS Theory is a constructivist theory concerned with mathematical learning whereas FCA is a mathematical theory itself. Together both theories provide different insights into how conceptual structures can be modelled and learned: FCA provides a model for a structural analysis of mathematical concepts and APOS Theory highlights the challenges involved in learning concepts that are complex and abstract.

1 Introduction

The question as to how a person learns mathematical concepts can be investigated from many different perspectives including cognitive and neuroscientific, mathematics educational, philosophical and structural considerations. For this paper we have selected two theories which provide very different views on what concepts are and how they may be learned, but which complement each other. The first theory, Formal Concept Analysis (FCA), was developed by Rudolf Wille in the 1980s as a mathematical lattice-based model of conceptual hierarchies with applications in data analysis and knowledge representation (Ganter & Wille, 1999). The second theory, APOS Theory (e.g., Dubinsky & McDonald (2002)), was developed by Ed Dubinsky in the area of mathematics education research based on a constructivist understanding of learning. The abbreviation APOS stands for Action, Process, Object and Schema and is explained further below.

Some parallels can be observed between how FCA and APOS Theory were invented. Both founders (Wille and Dubinsky) started out as pure mathematicians at about the same time and then developed a deep interest in teaching and learning. Both were influenced by pedagogical, philosophical theories (Peirce and Piaget, respectively). Both theories have mathematical constructs at their core (lattices in FCA and functions in APOS Theory) which correspond to the mathematical research interests of their

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founders (general algebra and functional analysis). Both attracted sufficient interest to each establish a research community that is still active today.

On the surface both theories are quite different. One theory (FCA) focuses on mathematical modelling which can be applied to educational data, but can also be used in many other domains. Its model of concepts is a mathematical abstraction of a philosophical understanding of what concepts are. Thus it is somewhat removed from cognitive and educational models of learning. The other theory (APOS Theory) is mainly concerned with what happens when students mentally construct concepts within an educational setting. Contrary to FCA, APOS Theory does not provide a formal, mathematical description of its notions. Nevertheless we argue in this paper that both theories are complementary to each other. Each provides viewpoints that are missing from the other theory, but might be beneficial to a more in-depth analysis of mathematical learning.

Section 2 briefly describes the two theories. It is followed, in Section 3 and Section 4, by a closer inspection of the transitions between the main stages suggested by APOS Theory and their relationship to FCA. The paper finishes with a conclusion.

2 A short description of the two theories

2.1 Formal Concept Analysis (FCA)

FCA is a theory of knowledge representation that presents a mathematical model for conceptual hierarchies using lattice theory (Ganter & Wille, 1999). It formalises notions of classification, ordering, hierarchies and concepts. Because FCA has been a topic of this conference for many years, this paper does not include an introduction to FCA. This section argues that a notion of 'formal concept' provides a means for modelling concepts occurring in formal disciplines, such as mathematics, and explaining how these differ from concepts occurring in the mental processing of natural language.

The definition of 'concept' in FCA is a formal mathematical definition and does therefore not necessarily express exactly the same as what researchers from other disciplines perceive as concepts. Nevertheless the idea of a concept consisting of an extension and an intension is consistent with a philosophical notion of 'concept' that has "grown during centuries from Greek philosophy to late Scholastic and has finally found its modern formulation in the 17th century by the Logic of Port Royal" (Mineau et al., 1999, p. 432). Thus FCA formalises a pre-existing notion.

The duality of extension and intension occurs in many mathematical disciplines. Functions can be represented extensionally by n-tuples of values and intensionally by formulas. In set theory, sets can be defined either extensionally by listing their values or intensionally via a condition. Boolean logical statements (such as $a \vee a = a$) can be evaluated by either writing truth tables or by applying transformations and axioms. In some domains, this duality leads to interesting questions. For example the field of abstract algebraic logic combines extensional questions about algebras with intensional questions about the dimension and expressive power of axiom bases. Interestingly there are some algebras which are not representable which means that they can be described intensionally, but not extensionally. Other algebras (such as proper relation algebras) are extensionally easy to describe, but do not have a finite axiom basis (or intensional description).

From a cognitive viewpoint, Endres et al. (2010) show that FCA provides a model of neural representations of stimuli within the visual cortex of the brain. Thus it is possible that concepts in the sense of FCA are highly relevant for modelling actual brain activity. In other disciplines concepts are often perceived as fuzzy, context-dependent, embodied or prototypical structures. For example, a definition of a concept for 'democracy' does not have a universal, precise extension and intension. Priss (2002) calls such concepts 'associative' and observes that the duality of formal versus associative does not just apply to concepts but to many structures. Formal concepts tend to occur in mathematics and natural sciences. For example, while there is no universal formal definition of 'bird' in natural languages, the concept of 'passerine bird' is formally defined in biology and has a precise extension and intension at any point in time (which can change, but only if new scientifically relevant facts are discovered). This notion of 'formal concept' is slightly broader than the one used in FCA because the extension and intension of 'passerine bird' are precise and finite yet impossible to be exhaustively listed.

With respect to mathematics education, Priss (2018) argues that one reason for why many people find it difficult to learn mathematical concepts is because such concepts are strictly formal in nature, but learners think of them in an associative manner. This distinction is somewhat akin to Tall & Vinner's (1981) distinction between concept definition (formal) and concept image (associative). For example, the mathematical definition of 'graph' corresponds to a formal concept. But if one asks students who have just started to learn mathematics at university to define 'graph' they might state that it is something that is represented graphically. Thus they are describing an associative concept via a prototypical feature. But 'being graphically represented' is neither necessary nor sufficient for a graph and thus irrelevant for the formal concept of 'graph'.

The conclusion of this section is that if mathematical concepts are formal in nature and have precise extensions and intensions, then FCA concept lattices can be used to structure mathematical knowledge. The content of such concept lattices would be incomplete because it is not possible to list all elements in the extensions and intensions, and not everything is known (as mentioned in the example of abstract algebraic logic above). But contrary to other FCA applications where concept lattices present an interpretation of data, mathematical knowledge can be represented with FCA so that the conceptual hierarchy in the lattices corresponds to provable mathematical statements. For example, a concept for 'relation' would be a superconcept of 'function' in the lattice which can also be proven with mathematical theory. The relationships in the lattice then represent structures that need to be learned by a student of mathematics.

2.2 APOS Theory

Dubinsky's APOS Theory (Dubinsky & McDonald, 2002) is based on Piaget's 'reflective abstraction' and states that mathematical knowledge is learned as a progression involving actions, processes, objects[1] and schemas. Actions consist of actually performing some transformation. For example, with respect to a function that produces the square of two numbers, an action-level understanding means that one can multiply

[1] In FCA the notion 'object' traditionally has a different meaning. In order to avoid confusion, in this paper we use 'element' instead of 'object' for the FCA notion.

a number with itself if shown how to do this. A process-level understanding of this function means that one can imagine or think about calculating squares without actually doing it. At this level one can calculate squares for actual numbers, but also for an unknown x or for complex expressions (for example $(a + b)^2$). Furthermore one understands that the function can be reversed (by obtaining a square root) and one can think about the behaviour of the function as it approaches infinity. Reaching a process-level understanding is thus a complex achievement. An object-level understanding of this function encapsulates the function itself into an object. This means that it can be composed with other functions and actions, processes and transformations can be applied to it. For example, a square function could be used as an input to another function.

A schema combines actions, processes and objects that belong together. Thus a schema of functions involves a general understanding of how they are used and how their actions, processes and objects relate to each other. Schemas can become objects themselves and can be combined with other schemas. But Arnon et al. (2013, p. 26) indicate that even within APOS Theory the notion of a schema is not completely finalised and more research about schemas is needed. A 'genetic decomposition' is a detailed description of a schema that shows dependencies between the mental constructions and thus can be used to determine the sequence in which the materials might be learned. Considering the conclusion of the last section about the possibilities of representing mathematical knowledge with FCA, the suggestion we are proposing in this paper is that FCA concept lattices could be helpful for representing the genetic decompositions of APOS Theory. Currently, genetic decompositions are designed by teachers based on experience and by using data from interviews with students. The representation of genetic decompositions is semi-formal and their relationships are a mixture of conceptual, part-whole and other relationships. Thus a genetic decomposition cannot be converted into a lattice. But we are proposing that lattices could be used as building blocks of genetic decompositions with other relationships possibly added to the lattices.

It should be emphasised that although the stages of APOS Theory indicate a progression, an individual does not always acquire these in this sequence. Sometimes a student might already know some process-level aspects while still being mostly in an action-level stage. Also a student can switch between and combine stages while working on a particular task. One important purpose of determining an APOS analysis of a particular mathematical topic is to devise a teaching and learning cycle using activities (which help students to make the required mental constructions), class discussions (during which a teacher observes if the students were successful at forming appropriate mental constructions and relates what has been learned to relevant mathematical knowledge) and exercises (which reinforce what has been learned and prepare students for the next iteration of the cycle). This is called the ACE learning cycle consisting of Activities, Class discussions and Exercises. The ACE learning cycle approach has been shown in numerous studies to be more effective than traditional lecture-centric teaching methods (Arnon et al., 2013).

This very brief description of APOS Theory is obviously incomplete (cf. Arnon et al. (2013) for a comprehensive discussion of APOS Theory). It is possible that APOS Theory is mainly only relevant for the learning of mathematics and similar formal domains. An example from music education below shows that the transition from action to process (called ‘interiorisation’) can occur in other domains. But the transition from process to object (called ‘encapsulation’) involves some form of abstraction and may only be relevant for domains where abstraction is frequently encountered. The next two sections further investigate interiorisation and encapsulation, respectively.

3 Interiorisation and conceptualisation

All core notions of APOS Theory are constructivist and focus on mental constructions. Nevertheless it is argued in this paper that it might be of interest to compare them to similar notions from other disciplines. For example, because interiorisation implies that internal structures are created and because, as mentioned above, experts can deliberately switch between different APOS stages, it should be remarked that it is also possible to consciously externalise some internal thought. Scaife and Rogers (1996) coined the notion of ‘external cognition’ in the context of understanding how graphical representations work. It refers to using external representations during cognitive processes. For example, mathematical tasks may become easier when they are conducted with pen and paper. External cognition implies that external representations may add a significant cognitive value by making something easier which would be difficult otherwise.

We propose considering interiorisation to be an example of conceptualisation because understanding a mathematical concept involves understanding which other concepts are equivalent or implied by the concept and which statements can be proven about the concept (including which elements are in its extension and which attributes in its intension). We argue that processes, objects and schemas correspond to formal mathematical concepts. Actions are not concepts because they are scripted procedures which do not require any understanding.

A further aspect of interiorisation is often the creation of a function. Dubinsky considers processes usually as functions and being able to write something as a function is a good indicator that a student has achieved a process-level understanding (cf. Arnon et al. 2013, p. 199) even though “a Process is only one part of a function”. Not every concept corresponds to a function, but in mathematics many concepts can be represented by functions. With respect to the fact that concepts in FCA consist of extensions and intensions this suggests that it might be of interest in some applications to represent intensions of FCA concepts by functions (e.g. algorithms or procedures) instead or in addition to lists of attributes.

An interesting question is whether APOS Theory is also relevant for other disciplines. We argue that while encapsulation is only relevant if the domain is sufficiently abstract, interiorisation might be observed in other domains. For example, in music education, one can observe that music learners at first often do not perceive rests at all. At an action-level understanding they might observe a rest by making a conscious effort to be silent for the required amount of time. A process-level understanding of musical rests involves perceiving rests as an integral part of music where errors with respect to rests produce the same sense of incorrectness as an imprecise pitch or a false articulation. Interiorisation then corresponds to a cognitive change of perception or conceptualisation from something that is consciously performed to something that is internally felt.

In this case the resulting process (or concept) is neither a mathematical function nor a formal concept.

4 Encapsulation and switching to a meta-level

The process/object transition (or encapsulation) has been extracted from APOS Theory and been used in other theories, such as by Hazzan (2003). As stated above, we believe that encapsulation always involves some form of abstraction. In FCA terminology it corresponds to a concept from one formal context becoming an element in another formal context and thus requiring a meta-level. In mathematics this happens frequently and is possibly unlimited. Mathematical category theory provides an example where even the basic elements are already encapsulated. An example from another discipline is the notion of 'emic units' in linguistics which arose from the observation that the distinction between certain units (such as phoneme and phone) also applies to other linguistic units (morpheme, grapheme, lexeme etc) and thus presents a general (meta-level) structure of semiotic systems. But the number of levels in non-mathematical domains is limited.

We argue in this section that encapsulation has cognitive, structural and systems-theoretical aspects. An example of cognitive aspects of encapsulation is chunking. Cognitive scientists consider grouping information so that it becomes easier to process and memorise an innate feature of human cognition. For example, Miller (1956) observes that humans can hold only about 7 items in short-term memory. Thus if one wants to memorise something that contains more than 7 items, it needs to be subdivided into groups of not more than 7 items each. Experiments have, for example, shown that expert chess players are able to memorise the configuration of an entire chess board because they chunk it into subgroups (Gobet et al., 2001). It is not necessary for the subgroups to form a unit that is meaningful apart from aiding as a memorisation task. Thus not every chunk becomes a meta-level object. But the human brain seems to have a tendency to form chunks or gestalts even if they are not really meaningful, such as seeing pictures in the clouds.

Part-whole relationships are an example of structural aspects of encapsulation. Linguistic analyses have shown that meronymic, or part-whole relationships, are core structures of language (Miller et al. 1990) and thus of human thinking. A large number of different types of linguistic part-whole relations have been identified in the literature (cf. Priss (1996) for some examples). In philosophy there is an entire discipline (mereology) dedicated to a theory of part-whole relationships. Encapsulation creates objects which are wholes consisting of parts, but not every part-whole relationship creates a meta-level. For example, the concepts of 'finger', 'hand' and 'limb' are all at the same level of abstraction even though a finger is part of a hand which is part of a body. But the concept 'body parts' is at a meta-level and has 'finger', 'hand' and 'limb' in its extension.

A meta-level object has parts, but the parts are not necessarily relevant to how the object behaves itself. This relates to emergence which is a systems-theoretical aspect of encapsulation. In cognitive science and related fields the notion of 'emergence' is used to describe objects or features that emerge from interactions among multiple elements in a system without having a simple relationship between the original elements and the emerging objects (Clark, 1997). There are many examples of emergent phenomena, such as flight patterns among birds, weather phenomena, crystalline structures and physical properties that occur at macroscopic scales, but not at microscopic scales. Often the initial elements follow simple rules. For example, migrating birds arrange their position with respect to the other flying birds using simple rules. In John Conway's game of life[2] there are a few simple rules about cell behaviour which cause complex patterns to emerge. In these cases, there is a causal connection between the original system with its elements and the emerging objects, but it is not possible to describe this connection using a simple input/output mapping (Clark, 1997).

A claim of this paper is that chunking and part-whole relationships are necessary, but not sufficient features of encapsulation. Emergence is often a consequence of encapsulation. Any mathematical definition of an object with some operations or properties gives rise to other objects and properties. The purpose of encapsulation is so that the resulting object can be used in further processes and transformations. By becoming an independent unit and interacting with other objects and processes, new features and properties can be observed which were not purposefully created during the encapsulation and which have nothing to do with the properties of the original processes. For example, even though multiplication of numbers is commutative, matrix multiplication is not. There is no intuitive transfer of properties from parts to whole.

5 Conclusion

This paper argues that FCA and APOS Theory complement each other with respect to analysing how mathematical concepts are learned. FCA provides a model for mathematical knowledge as concept lattices of formal concepts. This clarifies for example differences between part-whole and subconcept relationships and the transition to a meta-level when a concept from one context becomes an element in another formal context. Concept lattices can also explain why some concepts are more difficult to be learned than others: If a formal context is expanded by adding further new elements and attributes, it can happen that the new lattice is quite similar to the previous one if the new elements and attributes do not 'disturb' the previous connections. But, for example, if a new attribute applies to many old elements which did not have anything in common in the previous lattice, then the new lattice might be radically different. Other interesting research questions for FCA are whether it might be useful in some applications to represent intensions by functions and how the relationship between a lattice and a meta-level lattice which has as elements the concepts of the other lattice can be characterised.

APOS Theory emphasises the difficulties related to learning mathematics because of challenging conceptualisations (interiorisations), encapsulations which lead from one level of abstraction to a meta-level and the complexity of schemas that encompass mathematical knowledge. FCA could potentially be employed[3] to derive a more formal representation of genetic decompositions which represent schemas and structure the se-

[2] en.wikipedia.org/wiki/Conway's_Game_of_Life

[3] Possibly in combination with conceptual graphs (Sowa, 2008) and concept graphs (Wille, 2002).

quence in which mathematical concepts can be learned. Thus FCA and APOS Theory complement each other, and a combination of both might provide insights about mathematical learning which surpass the analytic capabilities of either theory by itself.

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