Rectangular Euler Diagrams and Order Theory

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Abstract. This paper discusses the relevance of order-theoretical properties, such as order dimension, for determining properties of Euler diagrams, such as whether a given poset can be represented with or without shading. The focus is on linear, tabular and rectangular Euler diagrams with shading and without split attributes and constructions with subdiagrams and embeddings. Euler diagrams are distinguished from geometric containment orders. Basic layout strategies are suggested.

1 Introduction

Euler diagrams are a well-known means for visualising sets and their partial orders (posets). Automated drawing of Euler diagrams, however, is still a difficult challenge (Alsallakh et al. 2016). Not all posets can be represented with all types of Euler diagrams. Thus it is desirable to determine properties of posets that affect which strategies of diagram construction and layout are suitable. This paper considers order-theoretical properties for such purposes.

Users tend to perceive Euler diagrams as more intuitive to read than Hasse[1] diagrams (Priss 2020), but Hasse diagrams are much easier to automatically draw with a computer due to a number of well-known layout algorithms (Dürrschnabel & Stumme 2021). Algorithms for Euler diagrams exist (Alsallakh et al. 2016), but there are still many open questions about construction and layout. Maybe Hasse diagrams are in some sense closer to order theory than Euler diagrams because they appear to depend less on the 2-dimensional geometrical space in which they are drawn than Euler diagrams.

Two negative results are presented in this paper: first, even though the order dimension has relevance for Euler diagrams, it is not as helpful in determining whether a diagram can be drawn or not as one might wish. Second, rectangular Euler diagrams are not geometric containment orders. The reason why these two results are of interest is because order theory is the mathematical foundation of both Hasse and Euler diagrams. Thus one might assume that questions about Euler diagrams can be easily reduced to order-theoretical questions which may have already been solved. In some respects that is true. In particular, Formal Concept Analysis (FCA) which models binary relations as

[1] The diagrams should not be named after Hasse because he did not invent them, but the name is widely established in the literature.

mathematical lattices using order theory (Ganter & Wille 1999) has developed a large repertoire of models and algorithms that are relevant for Euler diagrams. Nevertheless this paper argues that applying FCA, and order theory in general, to Euler diagrams is beneficial but not always sufficient.

All types of visualisations tend to have particular purposes and particular limits. Euler diagrams are effective even for large amounts of data, if it is possible to extract a small set of features (e.g. 5-10 features) which is then utilised to aggregate the data. Euler diagrams are particularly suitable for displaying implications and dependencies amongst features that relate to a small number of concepts, for example, for teaching mathematics (Priss 2023). In other applications, Euler diagrams might not be an appropriate visualisation.

More details about and an overview of Euler diagrams are provided by Rodgers (2014) and Alsallakh et al. (2016). This paper focuses on linear, tabular and rectangular Euler diagrams with shading and without split attributes. Linear Euler diagrams are considered to have superior usability by Chapman et al. (2014). Petersen (2010) investigates conditions for determining whether a linear Euler diagram exists, however, she uses a different terminology and does not refer to her diagrams as Euler diagrams. The advantages of rectangular Euler diagrams were independently discussed by Paetzold et al. (2023) and Priss (2023). Paetzold et al. (2023) present algorithms for automated drawing but they ignore whether or not zones should be shaded.

The following section defines Euler diagrams and related terminology. Section 3 and 4 discuss the relevance of order dimension, lattice theory and geometric containment orders. Section 5 elaborates on layout strategies. Section 6 concludes the paper.

2 Euler Diagrams

Venn, Euler and Hasse diagrams are different means for representing partially ordered sets (cf. Priss, 2020). In the following definitions, two sets A and O are used. The elements of A are called *attributes*, the elements of O are called *objects*. The powerset of a set S is denoted by $P(S)$ in this paper.

Definition 1. *A* zone set $Z(A)$ *over a set* A *is a subset of the powerset* $P(A)$*. Elements of* $Z(A)$ *are called* zones *and elements of* $P(A) \setminus Z(A)$ *are called* gaps.

The running example for this section is a zone set $Z^{\star}(A) = \{\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c, d\}\}$ over $A = \{a, b, c, d\}$ with gaps $\{d\}, \{a, d\}, \{a, c\}$ and so on. A challenge for a discussion about Euler diagrams is that an attribute contains zones or, dually, is contained in zones if a zone is defined as a set of attributes. The same holds for objects: they belong to a zone with its attributes or, dually, contain a set of attributes. It is important to always be aware of which of the two dual perspectives is currently employed. The notions of "attribute" and "object" are meant to differentiate the perspectives and to avoid confusion with any meta-level use of the word "set" in a discussion about Euler diagrams. In the running example, the attributes are considered sets of objects as follows: $a = \{2, 3, 6\}$, $b = \{3, 5, 6, 7\}$, $c = \{4, 5, 6\}$ and $d = \{6\}$ with $O = \{1, 2, 3, 4, 5, 6\}$.

As usual, a *partially ordered set (or poset)* $(A, \leq)$ is a binary relation that is reflexive, symmetric and transitive (i.e., $\forall(a, b, c \in A)\, a \leq a \wedge (a \leq b \wedge b \leq a \Rightarrow a = b) \wedge (a \leq b \wedge b \leq c \Rightarrow a \leq c)$). A *linear order* is a poset in which any two elements are comparable (i.e., $\forall(a, b \in A)\, a \leq b \vee b \leq a$). It is well-known from set theory that $(Z(A), \subseteq)$ is a poset. In the context of Euler diagrams, it is actually more natural to consider the dual of $(Z(A), \subseteq)$, in this paper denoted by $(Z(A), \leq)$ with $z_1 \leq z_2 :\Longleftrightarrow z_1 \supseteq z_2$.

Figure 1 shows different visualisations of the running example $(Z^\star(A), \leq)$. The Hasse diagram in 1C is drawn with a node for each element of $Z^\star(A)$ and objects attached to the lowest zone which they belong to. The other two types of diagrams in Fig. 1 which are formally defined below are Euler diagrams (1B and 1D) and a formal context (1A) which has objects as rows, attributes as columns and a cross if an object is associated with an attribute. The relationship between formal contexts and Hasse diagrams is explained by FCA and not further discussed in this paper. The equivalence between the Euler and Hasse diagrams in Fig. 1 should be visually deducible from the zones and their objects. The shaded region in the Euler diagrams is explained below.

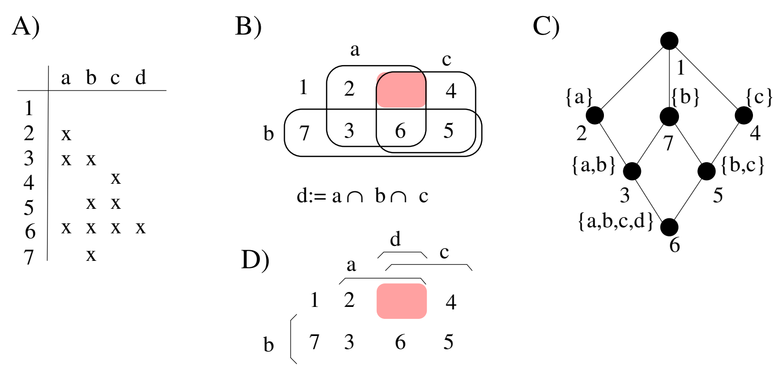

Fig. 1. $Z^\star(A)$ as A) formal context, B) and D) tabular Euler diagrams, C) Hasse diagram

Definition 2. *A* formal context (O, A, I) *is a binary relation* $I \subseteq O \times A$ *with two functions* $\phi_O : O \to P(A)$ *and* $\phi_A : A \to P(O)$ *with* $\phi_O(o) = \{a \mid a \in A, (o, a) \in I\}$ *and* $\phi_A(a) = \{o \mid o \in O, (o, a) \in I\}$.

The two functions determine which attributes are associated with which object and which objects are associated with which attribute. That means they read columns and rows from a formal context represented as a table as in Fig. 1, such as $\phi_A(\{a\}) = \{2, 3, 6\}$. The next definition connects zone sets with formal contexts.

Definition 3. *A zone set $Z(A)$ is represented as a formal context (O, A, I) if $\forall(o \in O)\, \phi_O(o) \in Z(A)$ and $\forall(z \in Z(A))\, \exists(o \in O)\, \phi_O(o) = z$. It is said that zone z* accommodates *all objects o for which $\phi_O(o) = z$ holds.*

In other words in a formal context, zones correspond to distinct rows. Each object is accommodated by exactly one zone and each zone accommodates at least one object. For example for $Z^\star(A)$, zone $\{a\}$ accommodates object 2. Because several rows of a formal context can have the same crosses, a zone can also accommodate more than one object. A gap $\{a, b, c\}$ is not a row of a formal context. Zones accommodate all objects that belong to an intersection of the zone's attributes without objects that are accommodated by lower zones according to $(Z(A), \leq)$. We distinguish between *contain* and *accommodate* in this paper: the intersection $a \cap b = \{3, 6\}$ contains 3 and 6. But zone $\{a, b\}$ accommodates only object 3 because 6 is accommodated by zone $\{a, b, c, d\}$. Even though the attributes of a formal context do not need to be sets in all applications, the function ϕ_A provides a natural means for interpreting them as sets. In general, FCA assumes no structural difference between objects and attributes and discusses many consequences arising from a dual relationship between objects and attributes. FCA usually completes partial orders to lattices. But that is not required for this paper. The following definitions establish Euler diagrams as used in Fig. 1 after first defining the types of graphical elements that are used.

Definition 4. *A* curve *is a Jordan curve (i.e. divides a plane into an interior and an exterior) that does not intersect itself. A* minimal region *is an area of a diagram that is enclosed by curve segments without containing any curve segments itself.* Shading *is a graphical darkening applied to minimal regions so that a minimal region can have exactly two states: either shaded or not shaded.*

Definition 5. *An* Euler diagram $E(Z(A))$ *is a graphical representation of* $Z(A)$ *where each* $a \in A$ *is represented as a curve and each zone* z *is represented by a single non-shaded minimal region that is contained in all curves* a *with* $a \in z$*. Shaded minimal regions represent gaps. Each gap corresponds to at most one shaded minimal region.* $\mathcal{E}$ *and* $\mathcal{E}^-$ *are the sets of all zone sets that can be represented as Euler diagrams with and without shading, respectively.*

Because of the dual perspectives, the condition $a \in z$ is correct because zones are sets of attributes even though graphically a zone is contained in all of the curves of its attributes. A similar definition can be constructed for *Euler diagrams with split attributes* by allowing an attribute to correspond to several curves. Unless stated differently below, in this paper the notion *Euler diagram* always refers to a diagram *without* split attributes.

Venn diagrams are special kinds of Euler diagrams that contain exactly as many minimal regions as the elements of a powerset $P(A)$. While Euler diagrams often have fewer shaded minimal regions than gaps, Venn diagrams have as many shaded minimal regions as gaps. Other authors use other definitions for Euler diagrams and discuss several conditions for when to consider Euler diagrams *well-formed* (Flower, Fish, & Howse 2008). In this paper, Definition 5 prohibits zones or gaps to be split into several minimal regions but otherwise does not require any further conditions.

Figure 1B shows an Euler diagram with 3 curves, 7 zones (including the outer zone which accommodates object 1) and a shaded minimal region for the gap $\{a, c\}$. Curves are allowed to be concurrent, but then they are drawn slightly dislocated as in Fig. 1B so that they can be visually distinguished. Priss (2023) argues that users will naturally regard the tiny extra regions that sometimes occur between dislocated concurrent curves as drawing errors and ignore them. Some concurrency can be avoided if curves that coincide exactly with a zone set are textually added, such as adding "$d := a \cap b \cap c$" in Fig. 1B instead of drawing a curve for d. Figure 1D displays the same information as 1B but uses a different notation. While Fig. 1B and 1D are structurally equivalent, it is often possible to represent a single zone set with structurally differing Euler diagrams. Therefore equivalence of diagrams is discussed further below. At a minimum, diagrams that are obtained from each other by changing sizes without changing relative locations should always be considered equivalent or identical. The next definition and lemma explain that shading allows to use the same layout of an Euler diagram for different zone sets.

Definition 6. *A* subdiagram *of an Euler diagram is obtained by shading 0 or more zones. A* superdiagram *of an Euler diagram is obtained by turning 0 or more shaded minimal regions into (non-shaded) zones. An* utmost diagram *of an Euler diagram is a superdiagram that does not contain any shaded minimal regions.*

Lemma 1. *I) Every Euler diagram has a unique utmost diagram.*
II) If $E_1(Z_1(A))$ is a subdiagram of $E(Z(A))$ then $Z_1(A) \subseteq Z(A)$.
III) If $Z_1(A) \subseteq Z(A)$, then $E_1(Z_1(A))$ need not be a subdiagram of $E(Z(A))$.

Proof. I) Because the utmost diagram is obtained by removing any shading which is a deterministic process. II) Shading turns zones into gaps and therefore reduces a set $Z(A)$ to a subset. III) Fig. 2 displays $Z^\star \setminus \{\{b\}\}$, but the Euler diagram in Fig. 2B is not a subdiagram of Fig. 1D because in both diagrams the arrangement of the zones is different (2 rows vs 1 row). The difference is not just a matter of shading. □

A)

	a	b	c	d
1				
2	x			
3	x	x		
4			x	
5		x	x	
6	x	x	x	x

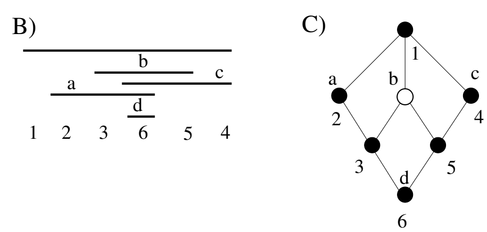

Fig. 2. For $Z^\star \setminus \{\{b\}\}$: A) Formal context, B) Linear Euler diagram, C) Hasse diagram

Fig. 2 shows an example where the corresponding Hasse diagram requires an element corresponding to gap ($\{b\}$), but the Euler diagram does not require a shaded minimal region. Priss (2020) calls elements of posets that occur but do not accommodate any objects *supplemental*. In Hasse diagrams, supplemental elements are drawn as non-filled circles and fulfil a similar role as shading in Euler diagrams. Priss (2020) provides a more detailed discussion of the drawability of Hasse diagrams with or without supplemental elements compared to Euler diagrams. It should be noted that in the Hasse diagrams in the remainder of this paper, only the attributes (and not the zones) are written into the diagrams. Euler diagrams tend to be easier to read if shading is avoided but that is not always possible:

Lemma 2. *I) All zone sets can be represented as Euler diagrams with shading.*
II) Not all zone sets can be represented as Euler diagrams without shading, i.e. $\mathcal{E}^- \subset \mathcal{E}$.

Proof. I) Venn diagrams are Euler diagrams according to Definition 5. Because Venn already showed that powersets of any size can be represented as Venn diagrams (Baron 1969) all subsets of powersets can be represented as subdiagrams of Venn diagrams. II) Depending on what shapes of curves are allowed, there are always zone sets that can only be represented with shading. The example in Fig. 1B cannot be represented without shading if the curves are rectangles. □

Hasse diagrams are equivalent to each other if they represent the same zone set and contain the same supplemental nodes or, visually, if they can be obtained from each other by moving nodes freely around as long as the upwards direction of the edges is not changed. For Euler diagrams it is not as easy to define equivalence because a zone set can be represented as a subdiagram of different utmost diagrams and it is not as clear what it means to transform one representation into another one. Thus, different kinds of equivalence can be defined. The least and most restrictive ones are:

Definition 7. *Two Euler diagrams* $E_1(Z_1(A))$ *and* $E_2(Z_2(A))$ *are* zone-equivalent *if* $Z_1(A) = Z_2(A)$ *and* shading-equivalent *if they are zone-equivalent and have the same utmost diagram.*

The following types of Euler diagrams are discussed in this paper:

Definition 8. *A* rectangular (Euler) diagram *is an Euler diagram where all curves are rectangles (ignoring the rounded corners) and all edges are either horizontal or vertical. A* tabular (Euler) diagram *is a rectangular diagram where all curves are placed into a single rectangle* t *so that for each curve (other than* t*) either the top and bottom edges are concurrent with* t *or the left and right edges are concurrent with* t*. A* linear (Euler) diagram *is a rectangular diagram where all curves are placed into a single rectangle* t *so that for all curves the top and bottom edges are concurrent with* t*.* $\mathcal{E}_1$*,* $\mathcal{E}_{1x1}$ *and* $\mathcal{E}_2$ *are the sets of all zone sets that can be represented as subdiagrams of linear, tabular and rectangular diagrams, respectively.*

An important question is to determine $\mathcal{E}_1$, $\mathcal{E}_{1x1}$ and $\mathcal{E}_2$. As mentioned in the introduction, Chapman et al. (2014) conclude that linear diagrams have superior usability compared to other Euler diagrams, but they do not investigate rectangular Euler diagrams. Priss (2023) argues that rectangular diagrams have similar usability as linear diagrams because users only need to trace a horizontal and a vertical direction. Priss (2021) suggests that tabular diagrams in the styles of Fig. 1B or 1D are optimal because they combine the advantages of linear diagrams with a more compact design. In particular Fig. 1D is just a table similar to many other kinds of tables that users are familiar with. Rectangular diagrams are furthermore easy to automatically process with a computer because computing curve intersections (as needed for shading) is straightforward for rectangles. Linear diagrams are uniquely identified by their linear sequence of zones. Tabular diagrams are uniquely identified by their sequence of zones for their rows and columns. Definition 8 requires that a single attribute (other than one containing everything) does not occur both as a row and as a column label for tabular diagrams because otherwise its shape would not be a rectangle. The relationship between linear and tabular diagrams can be described as a direct product.

Definition 9. *For zone sets $Z_1(A_1)$ and $Z_2(A_2)$ with $A_1 \cap A_2 = \emptyset$, the* direct product *is defined as a zone set $Z_1(A_1) \otimes Z_2(A_2) = \{z_1 \cup z_2 \mid z_1 \in Z_1(A_1), z_2 \in Z_2(A_2)\}$. A* direct product of two Euler diagrams *$E_1(Z_1(A_1))$ and $E_2(Z_2(A_2))$ with $A_1 \cap A_2 = \emptyset$ is an Euler diagram that is zone-equivalent to $Z_1(A_1) \otimes Z_2(A_2)$.*

Normally, direct products are Cartesian products, resulting in sets of tuples instead of sets of sets. But because the zone sets $Z_1(A)$ and $Z_2(A)$ do not have any attributes in common, the difference between sets and tuples is just a matter of notation. For Hasse diagrams FCA drawing algorithms exist that construct a direct product from its factors. For Euler diagrams such a graphical construction may not always be possible. But the direct product of two linear diagrams always yields a tabular diagram.

Lemma 3. *I) The utmost diagram of a tabular diagram is a direct product of two linear diagrams. Thus $\mathcal{E}_{1x1}$ is determined by direct products of elements of $\mathcal{E}_1$.*
II) A linear diagram with n attributes contains at most $2n$ zones.
III) A tabular diagram with n row attributes and m column attributes contains at most $4mn$ zones.

Proof. I) follows directly from Definition 8 because every cell of a table belongs to exactly one row and one column. II) because each curve splits at most one zone at each of its ends. III) because the linear diagram of the rows contains at most $2n$ and the linear diagram of the columns at most $2m$ zones, thus $2n2m = 4mn$. $\square$

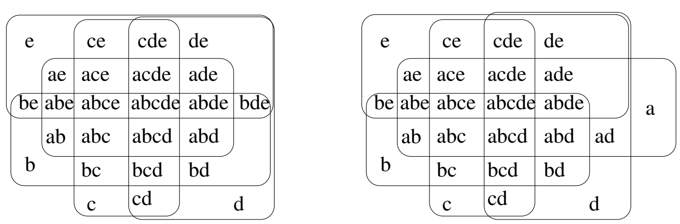

Fig. 3. Rectangular diagrams with 5 attributes with 26 zones (left) and 27 zones (right) including the outer zone. Brackets and commas are omitted for the zones, i.e. 'ce' instead of $\{c, e\}$.

It follows that a powerset with 5 attributes (i.e. $2^5 = 32$ elements) cannot be represented with a tabular diagram which would have at most $4 \times 2 \times 3 = 24$ zones. Figure 3 displays two rectangular diagrams with 5 attributes and slightly more than 24 zones. Rectangular diagrams are thus more powerful than tabular diagrams. But Fig. 3 also demonstrates that even small rectangular diagrams can already be difficult to read.

Venn diagrams without shading correspond to zone sets that are powersets without any gaps. Venn already showed that at most 3 attributes can be represented with circular curves (Baron 1969). A Venn diagram with at most 4 attributes can be represented with rectangular curves (as shown in Fig. 3 if the curve for a is deleted). For larger numbers of attributes the curves would need to be ellipses or concave shapes.

Lemma 4. *I) All zone sets with at most 4 attributes are in* $\mathcal{E}_{1x1}$.
II) If 3 attributes of a zone set each contain an object that the others do not contain and a zone containing all 3 attributes exists, then the zone set is not in $\mathcal{E}_1$.
III) If 5 attributes of a zone set each contain an object that the others do not contain and a zone containing all 5 attributes exists, then the zone set is not in $\mathcal{E}_{1x1}$.
IV) $\mathcal{E}_1 \subset \mathcal{E}_{1x1} \subset \mathcal{E}_2 \subset \mathcal{E}$.
V) For any number of attributes, there still exist some zone sets that are in $\mathcal{E}_2$.

Proof. I) Can be represented as a subdiagram of a Venn diagram for a powerset of 4 attributes. II) In a linear diagram, a zone $\{a, b\}$ must be between zones $\{a\}$ and $\{b\}$. For 3 attributes it is not possible to arrange them so that $\{a, b, c\}$ is between all three attributes without one of the attributes being contained in the union of the other attributes (as in Fig. 2). III) The utmost diagram would be a direct product of a linear diagram with 2 and one with 3 attributes. The latter is not possible according to II). IV) The inclusions $\subseteq$ are obvious. They are proper subsets because: $\mathcal{E}_1 \subset \mathcal{E}_{1x1}$ follows from II) and III). Fig. 3 shows examples that are in $\mathcal{E}_2$ but not in $\mathcal{E}_{1x1}$. A zone set that is a powerset of a set of 5 attributes is in $\mathcal{E}$ but not in $\mathcal{E}_2$. V) Fig. 8B below shows a rectangular diagram for a zone set with 6 attributes. The principle underlying that figure can be extended to any number of attributes. □

Euler diagrams can represent hundreds or thousands of objects if these are aggregated in some form. But Lemma 4 indicates that 5 attributes might already pose a problem. In order to increase the number of possible representations, Euler diagrams with split attributes can be defined. For example, the utmost diagram of Fig. 8C is a powerset of 6 attributes where 2 attributes (c and f) are split. It should be noted that in that case it is still required that every zone occurs at most once in a diagram. Thus a split attribute must be combined with other attributes in each instance in which it occurs.

Lemma 5. *If Euler diagrams are defined with split attributes (modifying Def. 5), then all zone sets can be drawn as tabular or even linear diagrams.*

Proof. Because the attributes can be split, even a linear diagram can be drawn where the zones occur in any random sequence and each column is labelled by all attributes that belong to a zone. □

Obviously, a random arrangement of zones as indicated in the proof is not useful because it will not render diagrams that are easy for humans to interpret. Furthermore, splitting attributes too many times certainly reduces readability. Therefore questions arise as to how to minimise shading in cases where a representation is possible and how to minimise splitting and shading otherwise. Criteria for determining whether representations are possible at all are essential. The next two sections discuss in how far order-theoretical notions might contribute to answering such questions. While splitting attributes may be necessary for applications, it is of interest to first investigate what is possible without splitting attributes as discussed in the next two sections.

3 Order dimension

Very "complex" zone sets cannot be represented as rectangular Euler diagrams. But it is not easy to characterise exactly how this complexity can be measured. One possible measure is the *order dimension* of a poset (e.g. Ganter & Wille 1999). First the notions of *order embedding* and *direct product* are required:

Definition 10. *For two posets $(P, \leq_p)$ and $(Q, \leq_q)$, a map $\psi : P \to Q$ is called an* order embedding *if $\forall (p, q \in P)\ p \leq_p q \iff \psi(p) \leq_q \psi(q)$.*
The direct product *of two posets $(P_1, \leq)$ and $(P_2, \leq)$ is a poset $(P_1 \times P_2, \leq)$ with $(x_1, x_2) \leq (y_1, y_2) \iff x_1 \leq y_1$ and $x_2 \leq y_2$.*

According to Ganter & Wille (1999) a finite poset $(P, \leq_p)$ can be order embedded into a powerset of a set and can be represented as a zone set using a formal context according to Def. 3. Thus at least for finite sets, every zone set is a poset and every poset can be modelled as a zone set. The direct product in Def. 9 then coincides with what is stated in Def. 10.

Definition 11. *A poset $(P, \leq)$ has an* order dimension *$dim(P, \leq) = n$ if $(P, \leq)$ can be embedded into a direct product of n linear orders and n is the smallest such number.*
A poset $(P, \leq)$ has a k-dimension *$dim_k(P, \leq) = n$ if $(P, \leq)$ can be embedded into a direct product of n linear orders of length k and n is the smallest such number.*

For example, Fig. 4A shows a direct product of 2 linear orders. Any poset that can be embedded into 4A has order dimension ≤ 2. The equivalent Euler diagram in Fig. 4B is constructed by rotating the Hasse diagram 45 degrees to the left and then drawing rectangles instead of edges. Embedding a poset (as in 4C) into another poset (such as 4A) entails arranging the nodes in similar positions and leaving sufficient edges so that the $\leq$-relation still holds between all nodes. As shown in 4D this may require some shading. Furthermore, contrary to a subdiagram which has the same attributes as its utmost diagram, *embedding may require different attributes*. In Fig. 4 the embedded poset has two attributes that are not in the original poset ($g := b \cap d$ and $h := f \cap a$). The Euler diagram in 4D is rectangular, not tabular. In this case, it can be represented as a tabular diagram (4E). The question arises as to which zone sets can be represented using embedding.

Definition 12. *Let $\mathcal{E}^+_{1x1}$ denote the set of zone sets that can be represented by embedding into tabular diagrams.*

Lemma 6. $\mathcal{E}^+_{1x1} = \mathcal{E}_2$

Proof. The vertical and horizontal curve segments of rectangular diagrams can be projected onto a vertical and a horizontal axis, respectively. The projections are linear diagrams. Every zone of a rectangular diagram is thus representable as embedded into a direct product of two linear diagrams. □

Thus embedding into $\mathcal{E}_{1x1}$ allows to represent as much as $\mathcal{E}_2$ whereas creating subdiagrams is less powerful because of $\mathcal{E}_{1x1} \subset \mathcal{E}_2$. But as mentioned above, in applications embedding is only suitable if it is acceptable to change attributes. An algorithm for drawing rectangular diagrams might consist of determining two linear diagrams in which the diagram can be embedded or which it is a subdiagram of.

The next lemma discusses the relevance of order dimension for Euler diagrams. For example, zone sets in $\mathcal{E}_1$ can have an order dimension larger than 1 as demonstrated by Fig. 2.

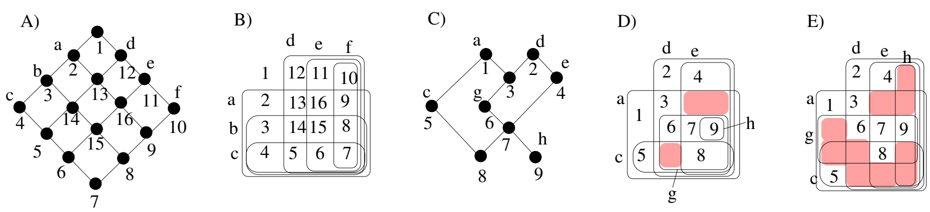

Fig. 4. Direct product of 2 linear orders and example of embedding

Lemma 7. *I)* $dim(P, \leq) = 1 \Rightarrow$ *zone set in* $\mathcal{E}_1 \Rightarrow dim(P, \leq) \leq 2$.
II) $dim(P, \leq) = 2 \Rightarrow$ *zone set in* $\mathcal{E}_2 \Rightarrow dim(P, \leq) \leq 4$.
III) Not every poset of $dim(P, \leq) = 2$ *is in* $\mathcal{E}_1$.
IV) Not every poset of $dim(P, \leq) = 2$ *is in* $\mathcal{E}_{1x1}$.
V) Not every poset of $dim(P, \leq) = 3$ *is in* $\mathcal{E}_2$.

Proof. I) If $dim = 1$ then the poset can be embedded into a linear order and is thus a linear diagram. If the poset is in $\mathcal{E}_1$ then it has a planar Hasse diagram and thus $dim \leq 2$. II) If $dim \leq 2$ then the construction mechanism described for Fig. 4 can be used. If the poset is in $\mathcal{E}_2$ then it can be represented by a rectangle containment order (as explained in the next section) for which it is known that $dim \leq 4$. III) For example, even if the linear orders in Fig. 4A were one element shorter, i.e. a 3×3 grid, then it cannot be represented as a linear diagram for the same reason as in Fig. 2. The node in the middle would have to be supplemental. IV) For example, if the linear orders in Fig. 4A were one element longer, i.e. a 5×5 grid, then it cannot be represented as a tabular diagram unless the node in the centre is supplemental. V) For example, a 3×3×3 grid cannot be represented as a rectangular diagram. Again, 26 zones can be represented, but the 27th node in the centre of the cube would have to be supplemental. □

The 2-dimension corresponds to the number of atoms of the smallest powerset which a poset can be embedded into.

Lemma 8. *I)* $dim_2(P, \leq) \leq 4 \Longrightarrow$ *zone set in* $\mathcal{E}_2$.
II) $dim_k(P, \leq) \leq 2 \Longrightarrow$ *zone set in* $\mathcal{E}_2$.

Proof. I) according to Lemma 4 powersets of 4 attributes are in $\mathcal{E}_{1x1}$ and $dim_2 = 4$ means an embedding into a powerset of 4 attributes. II) According to Lemma 7II). □

In summary, the order dimension of a poset provides some hints as to whether it might be representable as a linear, tabular or rectangular Euler diagram. Posets arising from applications, however, often have at least order dimension 3 or 4. And in those cases, it is not clear whether or not the poset can be represented by a rectangular or tabular diagram but Dürrschnabel & Priss (2024) provide some further criteria.

4 Lattices and Geometric Containment Orders

A finite poset is called a *lattice* if each set of elements has a supremum and an infimum. Priss (2020) explains that some zone sets of Euler diagrams only form lattices if supplemental nodes are added (e.g. Fig. 2C and Fig. 5). Supplemental nodes correspond to intersections of curves whose smallest larger container is a union. For example, $\{c, d\}$ in 5B is a gap, but $c \cap d$ is visible in the linear diagram even though it can only contain objects that are also in a or b, i.e. $c \cap d \subseteq a \cup b$. Furthermore, the empty set cannot be satisfactorily represented in Euler diagrams because it corresponds to every gap and could simultaneously be inserted everywhere in a diagram. In a Hasse diagram (as in 5A), the empty set can always be added as a supplemental node at the bottom if it is not already there. In applications the bottom element is often supplemental and can be omitted because it represents attributes which exclude each other. The Hasse diagrams in Fig. 5 demonstrate that with supplemental nodes both posets are lattices. Without supplemental nodes, some elements would not have suprema and infima. In fact, Dürrschnabel & Priss (2024) show that a zone set together with its supplemental nodes always forms a lattice. FCA provides algorithms for calculating such lattices.

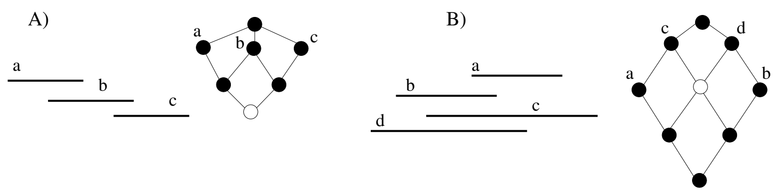

Fig. 5. Zone sets with supplemental nodes are always lattices

From the viewpoint of order theory, an important negative result is that geometric containment orders are not Euler diagrams. On first sight, it might seem as if interval containment orders are the same as linear diagrams and rectangle containment orders are the same as tabular diagrams. If that was the case, then obtaining rectangular Euler diagrams would be mostly trivial because it is a well-known fact that any poset of order dimension 2 (or less) can be represented by an interval containment order and any poset of order dimension 4 (or less) can be represented by a rectangle containment order and data from sets in applications usually have order dimension 4 (or less). But the bounds presented in Lemma 7 cannot be reduced which is an indication that Euler diagrams are not geometric containment orders.

Definition 13. *A* geometric containment order *is an inclusion order in an Euclidean space where* $x < y$ *if and only if the shape corresponding to x is properly included in the shape of y. An* interval containment order *is a geometric containment order of intervals on the real line. A* rectangle containment order *is a geometric containment order of rectangles on a real valued coordinate system with two axes.*

In an interval containment order, $b < a$ means that the left endpoint of a is smaller than the left endpoint of b and the right endpoint of a is larger than the right endpoint of b. The same principle is applied to the four cornerpoints of a rectangle in a rectangle containment order. Even though the interval containment order in Fig. 6A is the same picture as in Fig. 2B (apart from the added label u), the corresponding Hasse diagrams are different in the case of a linear Euler diagram and an interval containment order. The difference is that geometric containment orders only consider containment and incomparability, but not intersections and unions. Thus a, b and c in Fig. 6A are mutually incomparable. They are all contained in u and all contain d. But the fact that b is contained in the union of a and c is ignored. In other words, geometric containment orders only represent an ordering on attributes, but ignore objects. For example in Fig. 6A, it is not possible for $b \setminus (a \cup c)$ to accommodate any objects but that is irrelevant if the diagram is read as a geometric containment order.

Lemma 9. *Euler diagrams and geometric containment orders are not the same.*

Proof. As explained in the previous paragraph and because of Lemma 7. □

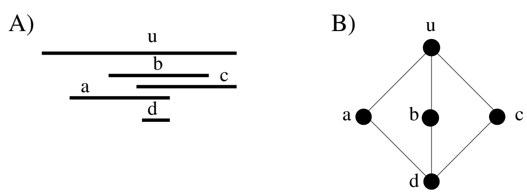

Fig. 6. A) Interval containment order and B) its Hasse diagram

Instead of embeddings into direct products of linear orders a different, more commonly used (and equivalent) definition of order dimension involves intersections of linear orders. In that case, order dimension 2 means that a poset can be represented as an intersection of two linear orders. For example, the interval containment order of Fig. 6 can be represented as an intersection of $d < c < b < a < u$ and $d < a < b < c < u$. In Fig. 7 these two linear orders are utilised to produce different graphical representations. Each diagram has two axes which are labelled with the attributes in a sequence according to the two linear orders. In 7A and 7B the axes form right angles, in 7C a 180 degree angle. A line then connects the two locations of each attribute resulting in rectangular curves in 7A1, triangular curves in 7A2 and B and linear curves in 7C. Essentially, Figure 7A1 follows the same principle as Fig. 4. Figure 7 demonstrates that different curve shapes produce different numbers of zones. Interestingly, if the curve shape is modified from rectangle to triangle (as in 7A2 and 7B), the line connecting b can be moved to either produce the Hasse diagram in 7A or in 7B whereas using rectangles only the Hasse diagram in 7A is possible. The figure also shows that posets of order dimension ≤ 2 (as in 7A1) can always be represented as interval containment orders (7C) requiring only to change the angle between the axes. But 7C contains fewer zones than 7A and cannot accommodate any objects in zone $\{b\}$.

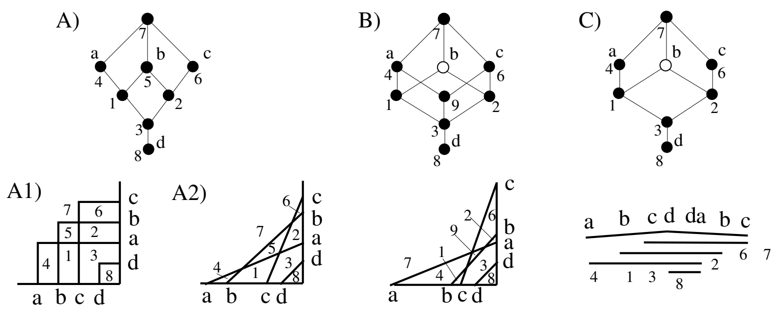

Fig. 7. Different Euler diagrams corresponding to the same interval containment order

5 Layout Strategies for Rectangular Euler Diagrams

A motivation for this research is to develop software for generating diagrams from databases using existing FCA algorithms but with Euler diagrams instead of Hasse diagrams. The aim is to automatically produce diagrams for very small data sets which are easy to read for users who have not been trained in FCA. This section does not intend to provide an overview of the literature on diagram design and layout but to sketch the design strategies that we have identified for our software. The usability of diagrammatic representations has been studied by many authors for many purposes but small changes can impact the results. For example, a study by Blake et al. (2014) derives 9 guidelines for good Euler diagrams one of which states that circles are preferable to rectangles. But contrary to their rectangles ours are dislocated and with rounded corners (adhering to their guideline of using smooth curves). Therefore Blake's negative result about rectangular diagrams is not applicable to our research.

An advantage of tabular diagrams is that users know how to read tables and only rows and columns matter. But if one compares the rectangular diagram in Fig. 4D with the tabular diagram in Fig. 4E, then the rectangular diagram is probably easier to read because it contains fewer shaded minimal regions. In this case the diagram in 4D is mostly tabular. Its two non-tabular rectangles (for g and h) do not extend across a large number of rows and columns and are thus not difficult to visually discern. Furthermore, the relationships $h < a$ and $g < d$ are clear in 3D and obscured in 3E. In summary, we have identified the following design strategies for obtaining diagrams with respect to our software:

S1 produce tabular diagrams if possible
S2 minimise shading
S3 try to visually preserve structures such as containment relationships and symmetries
S4 group shaded areas into large blocks instead of scattering them across the diagram
S5 avoid splitting attributes
S6 use added textual explanations if they simplify a diagram
S7 if the diagram is too large: reduce or split the data set

Unfortunately, some of the strategies are mutually contradicting. Fig. 8 presents six different Euler diagrams for the Hasse diagram in 8A. The universal set corresponding to the top node of the Hasse diagram is only drawn in 8D and 8E and omitted in the other diagrams. The embedding into a product of two linear orders in 8B follows the constructions in Fig. 4. A subdiagram of a powerset poset with 6 atoms is shown in 8C as a tabular diagram with two split attributes. Omitting rows and columns in 8C that are fully shaded results in 8F. An embedding into a powerset poset with 4 atoms is displayed in 8G. Figure 8G adheres to S2 and S5 by reducing shading and avoiding splitting but contradicts S3 because of scattered shaded minimal regions and difficult to see rectangles. The linear diagrams in 8D and 8E contradict S5 (split attributes) and S3 (containment and symmetries) but optimise S2. They contradict S3 in different ways: in 8D all attributes are structurally equal whereas 8E preserves one containment relationship. Usability experiments could determine which diagrams are most suitable for certain tasks. In this case, a tabular diagram without split attributes is not possible. Amongst the tabular diagrams, Fig. 8D appears to be least cluttered. The meaning of "a...f" in this diagram can be easily explained to users with an added short textual statement.

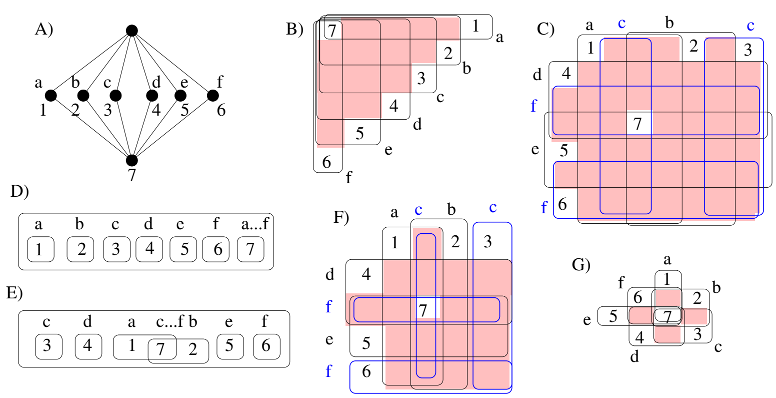

Fig. 8. Seven different representations of the same zone set

The examples in Fig. 8 are all subdiagrams of or embeddings into standard diagrams, such as products of linear orders or powersets which are easy to generate. The number of shaded minimal regions can be counted and compared. An FCA method called “conceptual partitioning” supports decisions about which attributes might be most suitable for splitting (Priss, 2023). The grouping of shaded areas can be optimised by permuting the sequence of the rows and columns. Last but not least, algorithms need not be fully automated. Drawing high quality Euler diagrams completely by hand is very time consuming. For the production of high quality diagrams, a software that provides a selection of possible layouts for additional manual modification with the support of a suitable drawing software is already very desirable.

6 Conclusion

This paper was written in the context of developing software for generating Euler diagrams with comfortable drawing capabilities so that the diagrams can be manually edited to some degree, but also automatically generated from binary relations. It builds on previous work by Priss (2020, 2021 and 2023) and continues with the goal of building software that visualises mathematical concepts for educational purposes. The currently existing educaJS software[2] only provides Hasse and Venn diagrams and is intended to also produce Euler diagrams.

Even for small data sets, automatically generating “good” diagrams is surprisingly difficult. A strategy from Section 5 is to mostly generate tabular Euler diagrams because rectangular shapes are geometrically easy to construct and users know how to read tables. Section 4 elaborates on order-theoretical properties that are more of mathematical relevance and may not directly support construction algorithms whereas the order-theoretical properties presented in Section 3 directly support decisions about whether or not tabular diagrams without split attributes are possible at all. An important step is to determine whether an attribute set can be partitioned into two sets so that a tabular diagram can be produced as a direct product of two linear diagrams. Algorithms for such tasks are described by Dürrschnabel & Priss (2024). If a diagram without split attributes does not exist then the data can be reduced or split, for example using “conceptual partitioning” as discussed by (Priss, 2023).

References

1. Alsallakh, B., Micallef, L., Aigner, W., Hauser, H., Miksch, S., Rodgers, P. (2016). The State-of-the-Art of Set Visualization. Computer Graphics Forum, 35, 1. p. 234-260.
2. Baron, M. E. (1969). A note on the historical development of logic diagrams: Leibniz, Euler and Venn. The mathematical gazette, 53, 384, p. 113-125.
3. Blake, A.; Stapleton, G,; Rodgers, P.; Cheek, L.; Howse, J. (2014). The Impact of Shape on the Perception of Euler Diagrams. In Dwyer et al., Proceedings of Diagrams 2014, LNCS 8578, Springer, p. 123-137.
4. Chapman, P., Stapleton, G., Rodgers, P., Micallef, L., Blake, A. (2014). Visualizing sets: An empirical comparison of diagram types. In: International Conference on Theory and Application of Diagrams, Springer, p. 146-160.
5. Dürrschnabel, D., Stumme, G. (2021). Force-Directed Layout of Order Diagrams Using Dimensional Reduction. In: Formal Concept Analysis: 16th International Conference, ICFCA 2021, Springer. p. 224-240.

[2] https://upriss.github.io/educaJS/

6. Dürrschnabel, D., Priss, U. (2024). Realizability of Rectangular Euler Diagrams. Available at: arXiv:2403.03801
7. Flower, J., Fish, A., Howse, J. (2008). Euler diagram generation. Journal of Visual Languages & Computing, 19, 6, p. 675-694.
8. Ganter, B., Wille, R. (1999). Formal Concept Analysis. Mathematical Foundations. Berlin-Heidelberg-New York: Springer.
9. Paetzold, P., Kehlbeck, R., Strobelt, H., Xue, Y., Storandt, S., Deussen, O. (2023). Recteuler: Visualizing intersecting sets using rectangles. Computer Graphics Forum 42.
10. Petersen, W. (2010). Linear Coding of Non-Linear Hierarchies: Revitalization of an Ancient Classification Method. In A. Fink et al., (eds.), Advances in Data Analysis, Data Handling and Business Intelligence, Springer, p. 307-316.
11. Priss, U. (2020). Set Visualisations with Euler and Hasse Diagrams. In: Graph Structures for Knowledge Representation and Reasoning, Springer. p. 72-83.
12. Priss, U. (2021). Visualising Lattices with Tabular Diagrams. In: Proceedings of DIAGRAMS21, LNAI 12909, Springer.
13. Priss, U. (2023). Representing Concept Lattices with Euler Diagrams. In International Conference on Formal Concept Analysis, Springer, p. 183-197.
14. Rodgers, P. (2014). A survey of Euler diagrams. Journal of Visual Languages & Computing, 25, 3, p. 134-155.