Predicate logic

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Outline

Predicate logic

Proofs

Artificial intelligence

Predicate logic

Propositional logic: "and, or, not" and variables.

Propositional logic does not have quantifiers:

"All poodles are dogs."
"There is at least one black swan."
"Only supervisors are allowed to fill in the form."
"No person can solve this problem."

These sentences cannot be expressed in propositional logic.

Predicate logic

Quantifiers: all, at least one

First order logic: quantifiers for individuals.
"All squirrels eat nuts."

Second order logic: quantifiers for predicates.
S = "squirrels eat nuts"
"All S involves chewing."

Third order logic: ...

Syntax and semantics

Predicate logic is a formal language (like a programming language) with rules for

syntax (i.e. how to write expressions) and

semantics (i.e. how to formalise the meaning of expressions).

Syntax: well-formed formulas

- **logical symbols**: and, or, not, all, at least one, brackets, variables, equality (=), true, false
- **predicate and function symbols**
  (for example, Cat(x) for "x is a Cat")
- **term**: variables and functions
  (for example, Cat(x))
- **formula**: any combination of terms and logical symbols
  (for example, "Cat(x) and Sleeps(x)")
- **sentence**: formulas without free variables
  (for example, "All x: Cat(x) and Sleeps(x)")

Semantics: meaning

The meaning of a term or formula is a set of elements. The meaning of a sentence is a truth value.

The function that maps a formula into a set of elements is called an **interpretation**.

An interpretation maps
an **intensional description** (formula/sentence) into
an **extensional description** (set or truth value).

Validity

A sentence is ...

- **satisfiable** if it is true under **at least one** interpretation.
- **valid** if it is true under **all** interpretations.
- **invalid** if it is false under **some** interpretation.
- **contradictory** if it is false under **all** interpretations.

Example: "All x: Cat(x) and Sleeps(x)"

If this is interpreted on an island which only has one cat that always sleeps, this is satisfiable.
Since not all cats in all interpretations always sleep, the sentence is not valid.

Exercises

Are these sentences satisfiable, valid, invalid or contradictory? If a sentence is satisfiable or invalid, provide an interpretation which makes it true (or false).

- $1 + 1 = 1$
- $A \cap B =$ not(not $A \cup$ not B)
- All x: ToBe(x) or not ToBe(x)

Axioms

Mathematical theories distinguish between **axioms** and **theorems**.

Axioms cannot be proven, but are accepted as facts.
Theorems can be proven from axioms by using logical inference.

An example of using axioms

A Boolean logic is a set with operators: and ($\wedge$), or ($\vee$), not ($\overline{\ }$), the elements "true" and "false" and the axioms:

associativity	$A \wedge (B \wedge C) = (A \wedge B) \wedge C$	$A \vee (B \vee C) = (A \vee B) \vee C$
commutativity	$A \wedge B = B \wedge A$	$A \vee B = B \vee A$
absorption	$A \vee (A \wedge B) = A$	$A \wedge (A \vee B) = A$
distributivity	$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$	$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$
complements	$A \vee \overline{A} = 1$	$A \wedge \overline{A} = 0$

De Morgan's law is a theorem which can be proven from the axioms.

Proving de Morgan's law

$$
\begin{aligned}
&\overline{A} \vee \overline{B} = (\overline{A} \vee \overline{B}) \wedge 1 \\
&= (\overline{A} \vee \overline{B}) \wedge ((A \wedge B) \vee \overline{A \wedge B}) \\
&= (\overline{A} \wedge A \wedge B) \vee (\overline{A} \wedge \overline{A \wedge B}) \vee (\overline{B} \wedge A \wedge B) \vee (\overline{B} \wedge \overline{A \wedge B}) \\
&= 0 \vee (\overline{A} \wedge \overline{A \wedge B}) \vee 0 \vee (\overline{B} \wedge \overline{A \wedge B}) \\
&= (\overline{A} \wedge \overline{A \wedge B}) \vee (\overline{B} \wedge \overline{A \wedge B}) \\
&= ((\overline{A} \vee \overline{B}) \wedge \overline{A \wedge B}) \\
&= ((\overline{A} \vee \overline{B}) \wedge \overline{A \wedge B}) \vee 0 \\
&= ((\overline{A} \vee \overline{B}) \wedge \overline{A \wedge B}) \vee (A \wedge B \wedge \overline{A \wedge B}) \\
&= \overline{A \wedge B} \wedge ((\overline{A} \vee \overline{B}) \vee (A \wedge B)) \\
&= \overline{A \wedge B} \wedge ((\overline{A} \vee \overline{B}) \vee (A \wedge B) \vee (A \wedge B)) \\
&= \overline{A \wedge B} \wedge (((\overline{A} \vee (A \wedge B)) \vee (\overline{B} \vee (A \wedge B))) \\
&= \overline{A \wedge B} \wedge (\overline{A} \vee B \vee \overline{B} \vee A) \\
&= \overline{A \wedge B} \wedge 1 = \overline{A \wedge B}
\end{aligned}
$$

Syllogisms

The ancient Greek philosopher Aristotle introduced logical **syllogisms**.

Premise 1: All humans are mortal.
Premise 2: Socrates is a human.
Conclusion: Therefore, Socrates is mortal.

Logical inference

- Law of excluded middle: either P is true or (not P) is true
- Principle of contradiction: P and (not P) cannot both be true

Examples:

⋆ Either you are currently sitting in this class room or not.
⋆ The gender of the baby is either male or female or unknown.
⋆ Either you are wearing shoes or sandals.
⋆ Are you wearing a shirt or a t-shirt? Could be both.

Logical inference

- Modus ponens: (P implies Q) and P is true $\Longrightarrow$ Q is true
- Modus tollens: (P implies Q) and Q is false $\Longrightarrow$ P is false
- Contradiction: P implies (Q is true and false) $\Longrightarrow$ P is false
- Contraposition: (not Q implies not P) $\Longrightarrow$ (P implies Q)

Other types of proofs used in mathematics:
mathematical induction, construction, exhaustion, visual proof, ...

Exercise: which of these inferences are valid?

The days are becoming longer.
The nights are becoming shorter if the days are becoming longer.
Hence, the nights are becoming shorter.

The earth is spherical implies that the moon is spherical.
The earth is not spherical.
Hence, the moon is not spherical.

The new people in the neighbourhood have a beautiful boat.
They also have a nice car.
Hence, they must be nice people.

All dogs are carnivorous.
Some animals are dogs.
Therefore, some animals are carnivorous.

Reasoning (Charles Sanders Peirce)

Induction: Every dog I know has ears. $\Longrightarrow$ Dogs have ears.
(from specific examples to general)

Deduction: Dogs bark. Bobby is a dog. $\Longrightarrow$ Bobby barks.
(from general to specific)

Abduction: Sherlock Holmes: the murderer was left-handed.
Smith is left-handed. $\Longrightarrow$ Smith is the murderer.
(based on shared attributes)

Exercises: induction, deduction or abduction?

Birds fly. Penguins are birds.
Hence, penguins fly.

All swans on this lake are white.
Hence, all swans are white.

Some dogs are animals.
Some cats are animals.
Hence, some cats are dogs.

Exercises: induction, deduction or abduction?

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Using deduction in unclear domains can lead to false results.
Induction and abduction always need to be treated with caution.

Artificial intelligence

- The term “artificial intelligence” was coined in 1956 by John McCarthy.
- An attempt to build computer software which has reasoning capabilities similar to humans.
- Processing of natural language.
- Understanding and modelling of spatial, temporal and social situations.
- Knowledge representation and problem solving.
- Many approaches: neural networks, evolutionary algorithms, robots, distributed devices, ...

Knowledge representation

Uses formal languages to represent knowledge.

- Conceptual structures
- Ontologies
- Knowledge bases
- Expert systems

Tasks for knowledge representation systems

- Acquisition: new information is integrated into the system
- Retrieval: existing information is retrieved, query answering
- Reasoning: logical inferences

Reasoning

- Is this concept an element of this class?
- Are concepts, relations and assertions satisfiable/valid?
- Is the knowledge base consistent (i.e. free of contradictions)?

Complexity: the reasoning tasks must not take too much time or computing resources.

Natural language can be ambiguous

They drank two cups of tea because they were warm.
They drank two cups of tea because they were cold.

Natural language can be ambiguous

They drank two cups of tea because they were warm.
They drank two cups of tea because they were cold.

Time flies like an arrow.
Fruit flies like a banana. (Groucho Marx)

$\Longrightarrow$ It is difficult to write software that extracts information from natural language or that translates between different languages.

Programming with logic

- Prolog: declarative, logic programming language
- Lisp: favourite language for artificial intelligence programming
- Smalltalk: object-oriented, dynamically typed, reflective language
- Description logics: knowledge representation languages
- OWL: web ontology language